THEORY OF FUZZY INTEGRATION
Algebra, fuzzy measure and fuzzy integration

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Abstract

This paper presents the theory of fuzzy integration by reviewing the fuzzy measure theory which is the basis of the fuzzy integration. This theory finds its application in the economic sciences, particularly in the theory of decision as aggregation operators, representations of preference relations and machine learning tools. Contrary to the theory of integration and Lebesgue measure, in this framework, the additivity property is not necessary.

Keywords: Fuzzy measure, Fuzzy integration, σ-algebra.

Résumé

Ce papier présente sommairement la théorie de l’intégrale floue en passant en revue la théorie de mesure floue. L’intégrale floue trouve quelques applications intéressantes dans les sciences économiques, en particulier dans la théorie des décisions, notamment comme des opérateurs d’agrégation des représentations, de relations de préférence, des outils d’apprentissage automatique et tant d’autres. Nous montrons également que, contrairement à l’intégrale de Lebesgue, la propriété d’additivité n’est plus nécessaire pour caractériser l’intégrale floue.

Mots-clés: Mesure floue, Intégrale floue, σ-algèbre.

1 Introduction

The integration and measure theory plays a very important role in economics, game theory or theory of decision1. It naturally has a role major in several disciplines, including probability theory and analysis of recursive methods in dynamic macroeconomics (Emone et al., 2013).

Let us mention that the concept of a fuzzy measure was introduced by Choquet2 in 1954. In his paper, the work originated from this problem, whose significance had been emphasized by M. Brelot and H. Cartan: “Is the interior Newtonian capacity of an arbitrary borelian subset X of the space $R^3$ equal

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1See e.g. Gilboa (1989); Grabisch (1996); Hougaard and Keiding (1996) among others
2Gustave Choquet (1915-2006) is a French mathematician. Well know for his theory of capacities, see Choquet (1954), and the Choquet integral. He worked mainly in the domains of Functional analysis, Topology, Potential theory and Measure theory.
to the exterior Newtonian capacity of \( X \). In 1974, Sugeno\(^3\) defined also a fuzzy measure in the sense of fuzzy integrals.

The concept fuzziness in contrast to fuzzy sets has been introduced by Lotfi Zadeh in 1965. Since that time, many branches of Mathematics invite the concept of fuzziness. Today, we have fuzzy topology, fuzzy algebra, fuzzy logic, fuzzy algorithmic, fuzzy Galois theory, fuzzy Geometry, fuzzy Graph, etc. And many disciplines of engineering have widely use the tools of fuzzy mathematics, because of their powerful representation of imprecision.

The mathematical expression fuzzy measures, such suggested by Sugeno is in contrast to fuzzy sets. A fuzzy measure on \( X \) is characterized by assigning the grade of certainty of “\( x \in A \)” to each subset \( A \) of \( X \), where \( x \) is an unknown element of \( X \) (Murofushi and Sugeno, 1989). Author such as Grabisch (1996), has proved how fuzzy measure theory can be used in decision making. Gilboa (1989) demonstrated how we can use the Choquet integral in inter-temporal preferences to model interaction between criteria in a flexible way.

This paper is organized as follows. Section 2 discusses the notions of Boolean Algebra and \( \sigma \)-algebra on sets so as to help the reader to understand the rest of the paper. Section 3 tackles a brief description of fuzzy measure with some basic properties. Section 5 shows the fuzzy integrals, especially Sugeno integral, and Choquet integral.

## 2 Boolean Algebra and \( \sigma \)-algebra on sets

Let \( X \) be a set. A collection \( \mathcal{F} \) of a set \( X \) is called a (Boolean) algebra of sets if:

1. If \( A, B \in \mathcal{F} \), then \( A \cup B \).
2. If \( A \in \mathcal{F} \), then \( \bar{A} \in \mathcal{F} \).

A (Boolean) algebra is called a \( \sigma \)-algebra on \( X \) if every union of a sequence \( \{A_i\}_{i=1}^{\infty} \) of members of \( \mathcal{F} \) is a members of \( \mathcal{F} \), i.e. \( \forall i = 1, 2, ... \)

\[
\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}
\]  

(1)

For example: \( \mathcal{P} \) and \( \{\emptyset, X\} \) are (Boolean) algebras

Let \((X, \mathcal{F}, m)\) be a measure space\(^4\). An Additive measure on a \( \sigma \)-algebra \( \mathcal{F} \), denoted \( m \) is a map \( m : \mathcal{F} \longrightarrow [0, \infty] \), such that:

1. \( m(\emptyset) = 0 \) when \( \emptyset \in \mathcal{F} \) (Vanishing at \( \emptyset \))
2. \( \{E_n\} \subset \mathcal{F} \), a countable sequence of pairwise disjoint subsets of \( \mathcal{F} \) \( \implies m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i) \)

For example: Lebesgue measure is an additive measure.

There are some kinds of measure that are not additive. These kinds of measures are called Fuzzy measures or Capacities.

## 3 Fuzzy measure

Let \( X \) be a set. A measure \( \mu \) is said to be fuzzy measure on a measurable space \((X, \mathcal{F})\) iff:

1. \( \mu(\emptyset) = 0 \) when \( \emptyset \in \mathcal{F} \) (Vanishing at \( \emptyset \))

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\(^3\)Michio Sugeno, Japanese engineer, born in 1940. Graduated in the Department of Physics of the University of Tokyo and holds a Dr. Eng. degree. Served the Tokyo Institute of Technology as Research Associate, Associate Professor and Professor from 1965 to 2000. Currently, Emeritus Professor at the Tokyo Institute of Technology.

\(^4\)For more information about measure space, see Emeone et al. (2013)
2. \( \forall E \in \mathcal{F}, F \in \mathcal{F} \) and \( E \subset F \implies \mu(E) \leq \mu(F) \) (Monotonicity)

3. \( E_n \subset \mathcal{F}, E_1 \subset E_2 \subset \ldots, \) and \( \bigcup_{i=1}^{\infty} E_i \in \mathcal{F} \) (Continuity from below)

4. \( E_n \subset \mathcal{F}, E_1 \supset E_2 \supset \ldots, \) and \( \bigcap_{i=1}^{\infty} E_i \in \mathcal{F} \) (Continuity from above)

We said that \( \mu \) is lower (upper) semicontinuous fuzzy measure on \((X, \mathcal{F})\) iff it satisfies (i), (ii) and (iii) (or (i), (ii) and (iv)), or we can call both of them simply semicontinuous measures.

These measure help to deteminate the fuzzy integrals.

**Properties.** For any \( E, F \in \mathcal{F} \). We have the following properties:

1. If \( \mu(E \cup F) = \mu(E) + \mu(F) \), for all \( A \cap F = \emptyset \), then \( \mu \) is said *additive*

2. If \( \mu(E \cup F) \geq \mu(E) + \mu(F) \), for all \( A \cap F = \emptyset \), then \( \mu \) is said *superadditive*

3. If \( \mu(E \cup F) \leq \mu(E) + \mu(F) \), for all \( A \cap F = \emptyset \), then \( \mu \) is said *subadditive*

4. If \( \mu(E \cup F) + \mu(E \cap F) \geq \mu(E) + \mu(F) \), then \( \mu \) is said *supermodular*

5. If \( \mu(E \cup F) + \mu(E \cap F) \leq \mu(E) + \mu(F) \), then \( \mu \) is said *submodular*

6. If \( |E| = |F| \implies \mu(E) = \mu(E) \), then \( \mu \) is said *symmetric*

7. If \( \mu(E) = 0 \) or \( \mu(1) \), then \( \mu \) is said *boolean*

### 4 Fuzzy integrals

Fuzzy integrals have been defined by different manners. In this paper, we are going to focus on the most usual.

**Definition (Sugeno Integral).** Let \((X, \mathcal{F})\) a measurable space, where \( A \in \mathcal{F} \) and \( f \in \mathcal{F} \). Let \( \mu : X \rightarrow [0, 1] \) be a fuzzy measure, the fuzzy integral of \( f \) on \( A \) respect to \( \mu \) is defined by:

\[
\int_{A} f \, d\mu = \max_{\alpha \in [0,1]} \left[ \min(\alpha, \mu(A \cap F_{\alpha})) \right]
\]  

(2)

We can adapt the formula (4) to create another type of integral over a fuzzy set \( \tilde{A} \). In this case, we will have:

\[
\int_{\tilde{A}} f \, d\mu = \int_{X} \min(\mu_{A}(x), \mu_{\tilde{A}}) d\mu
\]

(3)

where \( \mu_{A}(x) \) is the membership function of the fuzzy set \( A \); see Kamingu (2016).

Example: Let \( X = [0, 1], \mathcal{B} \) be the classe of Borel sets in \( X, \mu = m^{2} \), where \( m \) is the Lebesgue measure, \( f(x) = 1/2 \). We have:

\[
F_{\alpha} = \{ x | f(x) \geq \alpha \} = [2\alpha, 1]
\]

Since \( \Theta = [0, 1/2] \), we only need to consider \( \alpha \in [0, 1/2] \). So we have:

\[
\int_{A} f \, d\mu = \max_{\alpha \in [0,1/2]} \left[ \min(\alpha, \mu(F_{\alpha})) \right]
\]

\[
= \max_{\alpha \in [0,1/2]} \left[ \min(\alpha, (1 - 2\alpha)^{2}) \right]
\]

In the expression, \((1 - 2\alpha)^{2}\) is a decreasing continuous function of \( \alpha \) when \( \alpha \in [0, 1/2] \). Hence, the maximum will be attained at the point which is one of the solutions of the equation

\[
\alpha = (1 - 2\alpha)^{2}
\]

that is, at \( \alpha = 1/4 \). Therefore, we have:

\[
\int_{A} f \, d\mu = \frac{1}{4}
\]
Properties of Sugeno Integral (Sugeno, 1974). The following theorem gives the most elementary of the fuzzy integral.

1. If $\mu(A) = 0$, then $\int_A f \, d\mu = 0$ for any $f \in F$

2. If $\int_A f \, d\mu = 0$, then $\mu(A \cap x|f(x) > 0) = 0$

3. If $f_1 \leq f_2$, then $\int_A f_1 \, d\mu \leq \int_A f_2 \, d\mu$

4. $\int_A f \, d\mu = \int f \cdot \chi_A \, d\mu$, where $\chi_A$ is the characteristic function of $A$

5. $\int_A k \, d\mu = \min \mu(A)$ for any constant $k \in [0, \infty)$

6. $\int_A (f + k) \, d\mu \leq \int_A f \, d\mu + \int_A k \, d\mu$ for any constant $k \in [0, \infty)$

Definition (Choquet Integral). Let $(X, \mathcal{F})$ a measurable space, where $A \in \mathcal{F}$ and $f \in F$. Let $\mu : X \rightarrow [0, 1]$ be a fuzzy measure, the fuzzy integral of $f$ on $A$ respect to $\mu$.

$\forall x \in \mathbb{R}: \{t | f(t) \geq x\} \in \mathcal{F}$

Then, the Choquet’s Integral is given by:

$$\int f \, d\mu = \int_{-\infty}^{0} \mu(\{t | f(t) \geq x\}) \, dx + \int_{0}^{\infty} \mu(\{t | f(t) \geq x\}) \, dx$$

(4)

Note that the integrals on the right-hand side are the Riemann’s Integrals.

Properties of Choquet Integral. The following theorem gives the most elementary of the Choquet’s fuzzy integral.

1. If $f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$

2. If $f, g : X \rightarrow \mathbb{R}$ are comonotone functions, that is, if for all $t, t' \in X$, it holds that $(f(t) - f(t)) \geq 0$, then $\int f \, d\mu + \int g \, d\mu \geq \int (f + g) \, d\mu$

3. If $\mu$ is 2-alternative, then $\int f \, d\mu + \int g \, d\mu \geq \int (f + g) \, d\mu$

4. If $\mu$ is 2-monotone, then $\int f \, d\mu + \int g \, d\mu \leq \int (f + g) \, d\mu$

5 Conclusion

The challenge of this note is to present a simple way of understanding fuzzy integrals by recalling the fundamental concepts of fuzzy measure which is the basis of the fuzzy integrals. Then, I show that Riemann integral and Lebesgue integral different to fuzzy integrals mainly by the property of additivity. Finally, I examine the Choquet and Sugeno integrals, which are the best known and most used. Techniques of fuzzy measure theory and fuzzy integral theory are applied in various domains, such as: pattern recognition decision-making under uncertainty, information retrieval, large-systems control, management science, and others. In the next publications, I will propose a more explicit analysis about their applications.
References


